

# Upper Bounds on Syntactic Complexity of Left and Two-Sided Ideals<sup>\*</sup>

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**Abstract.** We solve two open problems concerning syntactic complexity. We prove that the cardinality of the syntactic semigroup of a left ideal or a suffix-closed language with  $n$  left quotients (that is, with state complexity  $n$ ) is at most  $n^{n-1} + n - 1$ , and that of a two-sided ideal or a factor-closed language is at most  $n^{n-2} + (n - 2)2^{n-2} + 1$ . Since these bounds are known to be reachable, this settles the problems.

**Keywords:** factor-closed, left ideal, regular language, suffix-closed, syntactic complexity, transition semigroup, two-sided ideal, upper bound

## 1 Introduction

The *syntactic complexity* [3] of a regular language is the size of its syntactic semigroup [4]. The *transition semigroup*  $T$  of a deterministic finite automaton (DFA)  $\mathcal{D}$  is the semigroup of transformations of the state set of  $\mathcal{D}$  generated by the transformations induced by the input letters of  $\mathcal{D}$ . The transition semigroup of a minimal DFA of a language  $L$  is isomorphic to the syntactic semigroup of  $L$  [4]; hence syntactic complexity is equal to the cardinality of  $T$ .

The number  $n$  of states of  $\mathcal{D}$  is known as the *state complexity* of the language [1, 5], and it is the same as the number of left quotients of the language. The *syntactic complexity of a class* of regular languages is the maximal syntactic complexity of languages in that class expressed as a function of  $n$ .

A *right ideal* (respectively, *left ideal*, *two-sided ideal*) is a non-empty language  $L$  over an alphabet  $\Sigma$  such that  $L = L\Sigma^*$  (respectively,  $L = \Sigma^*L$ ,  $L = \Sigma^*L\Sigma^*$ ). We are interested only in regular ideals; for reasons why they deserve to be studied see [2, Section 1]. Ideals appear in pattern matching. For example, if a *text* is a word  $w$  over some alphabet  $\Sigma$ , and a *pattern* is an arbitrary language  $L$  over  $\Sigma$ , then an occurrence of a pattern represented by  $L$  in text  $w$  is a triple  $(u, x, v)$  such that  $w = uxv$  and  $x$  is in  $L$ . Searching text  $w$  for words in  $L$  is

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equivalent to looking for prefixes of  $w$  that belong to the language  $\Sigma^*L$ , which is the left ideal generated by  $L$ .

The syntactic complexity of right ideals was proved to be  $n^{n-1}$  in [3]. The syntactic complexities of left and two-sided ideals were also examined in [3], where it was shown that  $n^{n-1} + n - 1$  and  $n^{n-2} + (n - 2)2^{n-2}$ , respectively, are lower bounds on these complexities, and it was conjectured that they are also upper bounds. In this paper we prove these conjectures.

If  $w = u xv$  for some  $u, v, x \in \Sigma^*$ , then  $v$  is a *suffix* of  $w$  and  $x$  is a *factor* of  $w$ . A suffix of  $w$  is also a factor of  $w$ . A language  $L$  is *suffix-closed* (respectively, *factor-closed*) if  $w \in L$  implies that every suffix (respectively, factor) of  $w$  is also in  $L$ . We are interested only in regular suffix- and factor-closed languages. Since every left (respectively, two-sided) ideal is the complement of a suffix-closed (respectively, factor-closed) language, and syntactic complexity is preserved by complementation, our theorems also apply to suffix- and factor-closed languages, but our proofs are given for left and two-sided ideals only.

## 2 Preliminaries

The *left quotient* or simply *quotient* of a regular language  $L$  by a word  $w$  is denoted by  $Lw$  and defined by  $Lw = \{x \mid wx \in L\}$ . A language is regular if and only if it has a finite number of quotients. The number of quotients of  $L$  is called its *quotient complexity*. We denote the set of quotients by  $K = \{K_0, \dots, K_{n-1}\}$ , where  $K_0 = L = L\varepsilon$  by convention. Each quotient  $K_i$  can be represented also as  $Lw_i$ , where  $w_i \in \Sigma^*$  is such that  $Lw_i = K_i$ .

A *deterministic finite automaton (DFA)* is a quintuple  $\mathcal{D} = (Q, \Sigma, \delta, q_0, F)$ , where  $Q$  is a finite non-empty set of *states*,  $\Sigma$  is a finite non-empty *alphabet*,  $\delta: Q \times \Sigma \rightarrow Q$  is the *transition function*,  $q_0 \in Q$  is the *initial state*, and  $F \subseteq Q$  is the set of *final states*.

The *quotient DFA* of a regular language  $L$  with  $n$  quotients is defined by  $\mathcal{D} = (K, \Sigma, \delta, K_0, F)$ , where  $\delta(K_i, w) = K_j$  if and only if  $K_i w = K_j$ , and  $F = \{K_i \mid \varepsilon \in K_i\}$ . To simplify the notation, we use the set  $Q = \{0, \dots, n-1\}$  of subscripts of quotients to denote the states of  $\mathcal{D}$ ; then  $\mathcal{D}$  is denoted by  $\mathcal{D} = (Q, \Sigma, \delta, 0, F)$ . The quotient corresponding to  $q \in Q$  is then  $K_q = \{w \mid \delta(q, w) \in F\}$ . The quotient  $K_0 = L$  is the *initial* quotient. A quotient is *final* if it contains  $\varepsilon$ . A state  $q$  is *empty* if its quotient  $K_q$  is empty.

The quotient DFA of  $L$  is isomorphic to each complete minimal DFA of  $L$ . The number of states in the quotient DFA of  $L$  (the quotient complexity of  $L$ ) is therefore equal to the state complexity of  $L$ .

In any DFA, each letter  $a \in \Sigma$  defines a transformation of the set  $Q$  of  $n$  states. Let  $\mathcal{T}_Q$  be the set of all  $n^n$  transformations of  $Q$ ; then  $\mathcal{T}_Q$  is a monoid under composition. The *identity* transformation  $\mathbf{1}$  maps each element to itself. For  $k \geq 2$ , a transformation (permutation)  $t$  of a set  $P = \{q_0, q_1, \dots, q_{k-1}\} \subseteq Q$  is a *k-cycle* if  $q_0 t = q_1, q_1 t = q_2, \dots, q_{k-2} t = q_{k-1}, q_{k-1} t = q_0$ . A *k-cycle* is denoted by  $(q_0, q_1, \dots, q_{k-1})$ . If a transformation  $t$  of  $Q$  acts like a *k-cycle* on some  $P \subseteq Q$ , we say that  $t$  has a *k-cycle*. A transformation has a *cycle* if it

has a  $k$ -cycle for some  $k \geq 2$ . A 2-cycle  $(q_0, q_1)$  is called a *transposition*. A transformation is *constant* if it maps all states to a single state  $q$ ; it is denoted by  $(Q \rightarrow q)$ . If  $w$  is a word of  $\Sigma^*$ , the fact that  $w$  induces transformation  $t$  is denoted by  $w: t$ . A transformation mapping  $i$  to  $q_i$  for  $i = 0, \dots, n-1$  is sometimes denoted by  $[q_0, \dots, q_{n-1}]$ .

### 3 Left Ideals

#### 3.1 Basic Properties

Let  $Q = \{0, \dots, n-1\}$ , let  $\mathcal{D}_n = (Q, \Sigma_{\mathcal{D}}, \delta_{\mathcal{D}}, 0, F)$  be a minimal DFA, and let  $T_n$  be its transition semigroup. Consider the sequence  $(0, 0t, 0t^2, \dots)$  of states obtained by applying transformation  $t \in T_n$  repeatedly, starting with the initial state. Since  $Q$  is finite, there must eventually be a repeated state, that is, there must exist  $i$  and  $j$  such that  $0, 0t, \dots, 0t^i, 0t^{i+1}, \dots, 0t^{j-1}$  are distinct, but  $0t^j = 0t^i$ ; the integer  $j-i$  is the *period* of  $t$ . If the period is 1,  $t$  is said to be *initially aperiodic*; then the sequence is  $0, 0t, \dots, 0t^{j-1} = 0t^j$ .

**Lemma 1 ([3]).** *If  $\mathcal{D}_n$  is a DFA of a left ideal, all the transformations in  $T_n$  are initially aperiodic, and no state of  $\mathcal{D}_n$  is empty.*

*Remark 1 ([2]).* A language  $L \subseteq \Sigma^*$  is a left ideal if and only if for all  $x, y \in \Sigma^*$ ,  $Ly \subseteq Lxy$ . Hence, if  $Lx \neq L$ , then  $L \subset Lx$  for any  $x \in \Sigma^+$ .

It is useful to restate this observation in terms of the states of  $\mathcal{D}_n$ . For DFA  $\mathcal{D}_n$  and states  $p, q \in Q$ , we write  $p \prec q$  if  $K_p \subset K_q$ .

*Remark 2.* A DFA  $\mathcal{D}_n$  is a minimal DFA of a left ideal if and only if for all  $s, t \in T_n \cup \{1\}$ ,  $0t \leq 0st$ . If  $0t \neq 0$ , then  $0 \prec 0t$  for any  $t \in T_n$ . Also, if  $r \in Q$  has a  $t$ -predecessor, that is, if there exists  $q \in Q$  such that  $qt = r$ , then  $0t \leq r$ . (This follows because  $q = 0s$  for some transformation  $s$  since  $q$  is reachable from 0; hence  $0 \leq q$  and  $0t \leq qt = r$ .) In particular, if  $r$  appears in a cycle of  $t$  or is a fixed point of  $t$ , then  $0t \leq r$ .

We consider chains of the form  $K_{i_1} \subset K_{i_2} \subset \dots \subset K_{i_h}$ , where the  $K_{i_j}$  are quotients of  $L$ . If  $L$  is a left ideal, the smallest element of any maximal-length chain is always  $L$ . Alternatively, we consider chains of states starting from 0 and strictly ordered by  $\prec$ .

**Proposition 1.** *For  $t \in T_n$  and  $p, q \in Q$ ,  $p \prec q$  implies  $pt \leq qt$ . If  $p \prec pt$ , then  $p \prec pt \prec \dots \prec pt^k = pt^{k+1}$  for some  $k \geq 1$ . Similarly,  $p \succ q$  implies  $pt \geq qt$ , and  $p \succ pt$  implies  $p \succ pt \succ \dots \succ pt^k = pt^{k+1}$  for some  $k \geq 1$ .*

It was proved in [3, Theorem 4, p. 124] that the transition semigroup of the following DFA of a left ideal meets the bound  $n^{n-1} + n - 1$ .

**Definition 1 (Witness: Left Ideals).** *For  $n \geq 3$ , we define the DFA  $\mathcal{W}_n = (Q, \Sigma_{\mathcal{W}}, \delta_{\mathcal{W}}, 0, \{n-1\})$ , where  $Q = \{0, \dots, n-1\}$ ,  $\Sigma_{\mathcal{W}} = \{a, b, c, d, e\}$ , and  $\delta_{\mathcal{W}}$  is defined by  $a: (1, \dots, n-1)$ ,  $b: (1, 2)$ ,  $c: (n-1 \rightarrow 1)$ ,  $d: (n-1 \rightarrow 0)$ , and  $e: (Q \rightarrow 1)$ . For  $n = 3$ ,  $a$  and  $b$  coincide, and we can use  $\Sigma_{\mathcal{W}} = \{b, c, d, e\}$ .*

*Remark 3.* In  $\mathcal{W}_n$ , the transformations induced by  $a$ ,  $b$ , and  $c$  restricted to  $Q \setminus \{0\}$  generate all the transformations of the last  $n - 1$  states. Together with the transformation of  $d$ , they generate all transformations of  $Q$  that fix 0. To see this, consider any transformation  $t$  that fixes 0. If some states from  $\{1, \dots, n - 1\}$  are mapped to 0 by  $t$ , we can map them first to  $n - 1$  and  $n - 1$  to one of them by the transformations of  $a$ ,  $b$ , and  $c$ , and then map  $n - 1$  to 0 by the transformation of  $d$ . Also the words of the form  $ea^i$  for  $i \in \{0, \dots, n - 2\}$  induce constant transformations ( $Q \rightarrow i + 1$ ). Hence the transition semigroup of  $\mathcal{W}_n$  contains all the constant transformations.

*Example 1.* One verifies that the maximal-length chains of quotients in  $\mathcal{W}_n$  have length 2. On the other hand, for  $n \geq 2$ , let  $\Sigma = \{a, b\}$  and let  $L = \Sigma^* a^{n-1}$ . Then  $L$  has  $n$  quotients and the maximal-length chains are of length  $n$ .

### 3.2 Upper Bound

Our main result of this section shows that the lower bound  $n^{n-1} + n - 1$  is also an upper bound. Our approach is as follows: We consider a minimal DFA  $\mathcal{D}_n = (Q, \Sigma_{\mathcal{D}}, \delta_{\mathcal{D}}, 0, F)$ , where  $Q = \{0, \dots, n - 1\}$ , of an arbitrary left ideal with  $n$  quotients and let  $T_n$  be the transition semigroup of  $\mathcal{D}_n$ . We also deal with the witness DFA  $\mathcal{W}_n = (Q, \Sigma_{\mathcal{W}}, \delta_{\mathcal{W}}, 0, \{n - 1\})$  of Definition 1 that has the same state set as  $\mathcal{D}_n$  and whose transition semigroup is  $S_n$ . We shall show that there is an injective mapping  $f: T_n \rightarrow S_n$ , and this will prove that  $|T_n| \leq |S_n|$ .

*Remark 4.* If  $n = 1$ , the only left ideal is  $\Sigma^*$  and the transition semigroup of its minimal DFA satisfies the bound  $1^0 + 1 - 1 = 1$ . If  $n = 2$ , there are only three allowed transformations, since the transposition  $(0, 1)$  is not initially aperiodic and so is ruled out by Lemma 1. Thus the bound  $2^1 + 2 - 1 = 3$  holds.

**Lemma 2.** *If  $n \geq 3$  and a maximal-length chain in  $\mathcal{D}_n$  strictly ordered by  $\prec$  has length 2, then  $|T_n| \leq n^{n-1} + n - 1$  and  $T_n$  is a subsemigroup of  $S_n$ .*

*Proof.* Consider an arbitrary transformation  $t \in T_n$  and let  $p = 0t$ . If  $p = 0$ , then any state other than 0 can possibly be mapped by  $t$  to any one of the  $n$  states; hence there are at most  $n^{n-1}$  such transformations. All of these transformations are in  $S_n$  by Remark 3.

If  $p \neq 0$ , then  $0 \prec p$ . Consider any state  $q \notin \{0, p\}$ ; by Remark 2,  $p \preceq qt$ . If  $p \neq qt$ , then  $p \prec qt$ . But then we have the chain  $0 \prec p \prec qt$  of length 3, contradicting our assumption. Hence we must have  $p = qt$ , and so  $t$  is the constant transformation  $t = (Q \rightarrow p)$ . Since  $p$  can be any one of the  $n - 1$  states other than 0, we have at most  $n - 1$  such transformations. Since all of these transformations are in  $S_n$  by Remark 3,  $T_n$  is a subsemigroup of  $S_n$ .  $\square$

**Theorem 1 (Left Ideals, Suffix-Closed Languages).** *If  $n \geq 3$  and  $L$  is a left ideal or a suffix-closed language with  $n$  quotients, then its syntactic complexity is less than or equal to  $n^{n-1} + n - 1$ .*

*Proof.* It suffices to prove the result for left ideals. For a transformation  $t \in T_n$ , consider the following cases:

**Case 1:**  $t \in S_n$ .

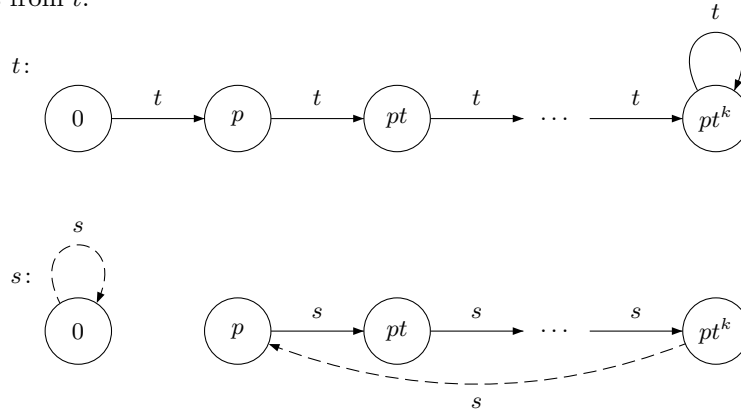
Let  $f(t) = t$ ; obviously  $f(t)$  is injective.

**Case 2:**  $t \notin S_n$  and  $0t^2 \neq 0t$ .

Note that  $t \notin S_n$  implies  $0t \neq 0$  by Remark 3. Let  $0t = p$ . We have  $p = 0t \prec 0tt = pt$  by Remark 2. Let  $p \prec \dots \prec pt^k = pt^{k+1}$  be the chain defined from  $p$ ; this chain is of length at least 2. Let  $f(t) = s$ , where  $s$  is the transformation defined by

$$0s = 0, \quad pt^k s = p, \quad qs = qt \text{ for the other states } q \in Q.$$

Transformation  $s$  is shown in Figure 1, where the dashed transitions show how  $s$  differs from  $t$ .



**Fig. 1.** Case 2 in the proof of Theorem 1.

By Remark 3,  $s \in S_n$ . However,  $s \notin T_n$ , as it contains the cycle  $(p, \dots, pt^k)$  with states strictly ordered by  $\prec$  in DFA  $\mathcal{D}_n$ , which contradicts Proposition 1. Since  $s \notin T_n$ , it is distinct from the transformations defined in Case 1.

In going from  $t$  to  $s$ , we have added one transition ( $0s = 0$ ) that is a fixed point, and one ( $pt^k s = p$ ) that is not. Since only one non-fixed-point transition has been added, there can be only one cycle in  $s$  with states strictly ordered by  $\prec$ . Since 0 can't appear in this cycle,  $p$  is its smallest element with respect to  $\prec$ .

Suppose now that  $t' \neq t$  is another transformation that satisfies Case 2, that is,  $0t' = p' \neq 0$  and  $p't' \neq p'$ ; we shall show that  $f(t) \neq f(t')$ . Define  $s'$  for  $t'$  as  $s$  was defined for  $t$ . For a contradiction, assume  $s = f(t) = f(t') = s'$ .

Like  $s$ ,  $s'$  contains only one cycle strictly ordered by  $\prec$ , and  $p'$  is its smallest element. Since we have assumed that  $s = s'$ , we must have  $p = 0t = 0t' = p'$  and the cycles in  $s$  and  $s'$  must be identical. In particular,  $pt^k t = pt^k = p(t')^k t' = p(t')^k$ . For  $q$  of  $Q \setminus \{0, pt^k\}$ , we have  $qt = qs = qs' = qt'$ . Hence  $t = t'$ —a contradiction. Therefore  $t \neq t'$  implies  $f(t) \neq f(t')$ .

**Case 3:**  $t \notin S_n$  and  $0t^2 = 0t$ .

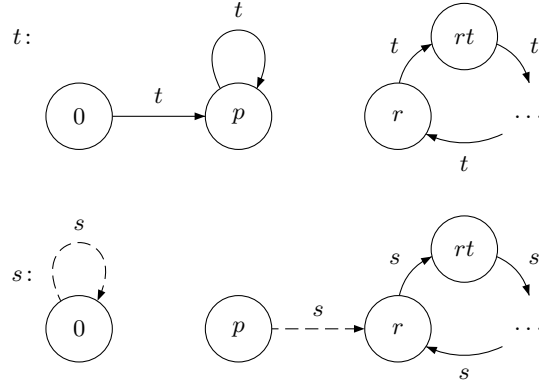
As before, let  $0t = p$ . Consider any state  $q \notin \{0, p\}$ ; then  $0 \prec q$  by Remark 2

and  $0t \preceq qt$  by Proposition 1. Thus either  $p \prec qt$ , or  $p = qt$ . We consider the following sub-cases:

- **(a):**  $t$  has a cycle.

Since  $t$  has a cycle, take a state  $r$  from the cycle; then  $r$  and  $rt$  are not comparable under  $\preceq$  by Proposition 1, and  $p \prec r$  by Remark 2. Let  $f(t) = s$ , where  $s$  is the transformation shown in Figure 2 and defined by

$$0s = 0, \quad ps = r, \quad qs = qt \text{ for the other states } q \in Q.$$



**Fig. 2.** Case 3(a) in the proof of Theorem 1.

By Remark 3,  $s \in S_n$ . Suppose that  $s \in T_n$ ; since  $p \prec r$ , we have  $r = ps \preceq rs = rt$  by the definition of  $s$  and Proposition 1; this contradicts that  $r$  and  $rt$  are not comparable. Hence  $s \notin T_n$ , and so  $s$  is distinct from the transformations of Case 1.

We claim that  $p$  is not in a cycle of  $s$ ; this cycle would have to be

$$p \xrightarrow{s} r \xrightarrow{s} rt \xrightarrow{s} \dots \xrightarrow{s} rt^{k-1} \xrightarrow{s} p, \text{ that is, } p \xrightarrow{s} r \xrightarrow{t} rt \xrightarrow{t} \dots \xrightarrow{t} rt^{k-1} \xrightarrow{t} p,$$

for some  $k \geq 2$  because  $r \neq p = pt$  and  $rt \neq p$ . Since  $p \prec r$  we have  $p \prec rt$ ; but then we have a chain  $p \prec rt \prec \dots \prec rt^k = p$ , contradicting Proposition 1.

Since  $p$  is not in a cycle of  $s$ , it follows that  $s$  does not contain a cycle with states strictly ordered by  $\prec$ , as such a cycle would also be in  $t$ . So  $s$  is distinct from the transformations of Case 2.

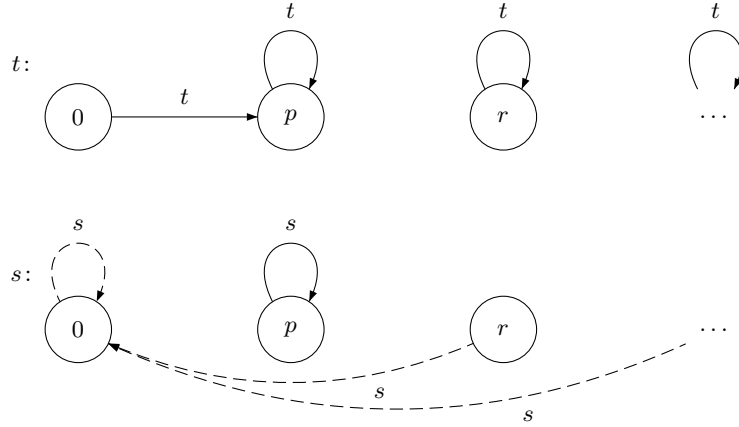
We claim there is a unique state  $q$  such that (a)  $0 \prec q \prec qs$ , (b)  $qs \not\preceq qs^2$ . First we show that  $p$  satisfies these conditions: (a) holds because  $ps = r$  and  $p \prec r$ ; (b) holds because  $ps = r$ ,  $ps^2 = rt$  and  $r$  and  $rt$  are not comparable. Now suppose that  $q$  satisfies the two conditions, but  $q \neq p$ . Note that  $qs \neq p$ , because  $qs = p$  implies  $qs = p \prec r = qs^2$ , contradicting (b). Since  $q, qs \notin \{0, p\}$ , we have  $qt = qs \not\preceq qs^2 = qt^2$ . But Proposition 1 for  $q \prec qt$  implies that  $qt \preceq qt^2$ —a contradiction. Thus  $p$  is the only state satisfying these conditions.

If  $t' \neq t$  is another transformation satisfying the conditions of this case, we define  $s'$  like  $s$ . Suppose that  $s = f(t) = f(t') = s'$ . Since both  $s$  and  $s'$  contain a unique state  $p$  satisfying the two conditions above, we have  $0t = 0t' = p$  and  $pt = pt' = p$ . Since the other states are mapped by  $s$  exactly as by  $t$  and  $t'$ , we have  $t = t'$ .

- **(b):**  $t$  has no cycles and has a fixed point  $r \neq p$ .

Because  $0 \prec r$  by Remark 2,  $0t \preceq rt$  by Proposition 1. If  $r$  is a fixed point of  $t$ , then  $p = 0t \preceq rt = r$ . Since  $r \neq p$ , we have  $p \prec r$ . Let  $f(t) = s$ , where  $s$  is the transformation shown in Figure 3 and defined by

$$\begin{aligned} 0s &= 0, & qs &= 0 \text{ for each fixed point } q \neq p, \\ qs &= qt \text{ for the other states } q \in Q. \end{aligned}$$



**Fig. 3.** Case 3(b) in the proof of Theorem 1.

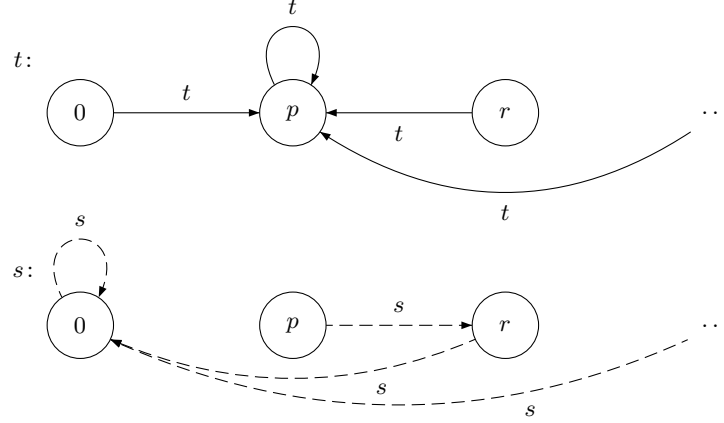
By Remark 3,  $s \in S_n$ . Suppose that  $s \in T_n$ ; because  $p \prec r$ ,  $ps = p$ ,  $rs = 0$ , and  $ps \preceq rs$  by Proposition 1, we have  $p \prec 0$ , which is a contradiction. Hence  $s$  is not in  $T_n$  and so is distinct from the transformations of Case 1. Also,  $s$  maps at least one state other than  $0$  to  $0$ , and so is distinct from the transformations of Case 2 and also from the transformations of Case 3(a).

If  $t' \neq t$  is another transformation satisfying the conditions of this case, we define  $s'$  like  $s$ . Now suppose that  $s = f(t) = f(t') = s'$ . There is only one fixed point of  $s$  other than  $0$  ( $ps = p$ ), and only one fixed point of  $s'$  other than  $0$  ( $p's' = p'$ ); hence  $0t = p = p' = 0t'$ . By the definition of  $s$ , for each state  $q \neq 0$  such that  $qs = 0$ , we have  $qt = q$ . Similarly, for each state  $q \neq 0$  such that  $qs' = 0$ , we have  $qt' = q$ . Hence  $t$  and  $t'$  agree on these states. Since the remaining states are mapped by  $s$  exactly as they are mapped by  $t$  and  $t'$ , we have  $t = t'$ . Thus we have proved that  $t \neq t'$  implies  $f(t) \neq f(t')$ .

- **(c):**  $t$  has no cycles, has no fixed point  $r \neq p$  and there is a state  $r$  such that  $p \prec r$  with  $rt = p$ .

Let  $f(t) = s$ , where  $s$  is the transformation shown in Figure 4 and defined by

$0s = 0$ ,  $ps = r$ ,  $qs = 0$  for each  $q \succ p$  such that  $qt = p$ ,  
 $qs = qt$  for the other states  $q \in Q$ .



**Fig. 4.** Case 3(c) in the proof of Theorem 1.

By Remark 3,  $s \in S_n$ . Suppose that  $s \in T_n$ ; because  $p \prec r$ ,  $ps = r$ ,  $rs = 0$ , and  $r = ps \preceq rs = 0$  by Proposition 1, we have  $r \prec 0$ —a contradiction. Hence  $s \notin T_n$  and  $s$  is distinct from the transformations of Case 1.

Because  $s$  maps at least one state other than 0 to 0 ( $rs = 0$ ), it is distinct from the transformations of Case 2 and 3(a). Also  $s$  does not have a fixed point other than 0, while the transformations of Case 3(b) have such a fixed point.

We claim that there is a unique state  $q$  such that (a)  $0 \prec q \prec qs$  and (b)  $qs^2 = 0$ . First we show that  $p$  satisfies these conditions. By assumption  $0 \prec p \prec r$  and  $rt = p$ ; also  $rs = 0$  by the definition of  $s$ . Condition (a) holds because  $0 \prec p \prec r = ps$ , and (b) holds because  $0 = rs = ps^2$ .

Now suppose that  $0 \prec q \prec qs$ ,  $qs^2 = 0$  and  $q \neq p$ . Since  $qs \neq 0$ , we have  $qs = qt$  by the definition of  $s$ . Because  $qt$  has a  $t$ -predecessor,  $p \preceq qt$  by Remark 2. Also  $qt = qs \neq p$ , for  $qs = p$  implies  $0 = qs^2 = ps = r$ —a contradiction. Hence  $p \prec qt$ . From  $qt = qs$  and  $q \prec qs$ , we have  $q \prec qt$ . Since  $qs^2 = 0$  we have  $(qt)s = 0$  and so  $(qt)t = p$ , by the definition of  $s$ . By Proposition 1, from  $q \prec qt$  we have  $qt \preceq (qt)t = p$ , contradicting  $p \prec qt$ . So  $q = p$ .

If  $t' \neq t$  is another transformation satisfying the conditions of this case, we define  $s'$  like  $s$ . Suppose that  $s = f(t) = t(t') = s'$ . Since  $s$  and  $s'$  contain a unique state  $p$  satisfying the two conditions above, we have  $0t = 0t' = p$  and  $pt = pt' = p$ . Then  $r$  and the states  $q \succ p$  with  $qt = p$  are determined by  $p$ , since they are precisely the states  $q \succ p$  with  $qs = 0$ . Since the other states are mapped by  $s$  exactly as by  $t$  and  $t'$ , we have  $t = t'$ , and  $f$  is again injective.

• **All cases are covered:**

Now we need to ensure that any transformation  $t$  fits in at least one case. It is clear that  $t$  fits in Case 1 or 2 or 3. For Case 3, it is sufficient to show that if



(i)  $t \notin S_n$  does not contain a fixed point  $r \neq p$ , and (ii) there is no state  $r$  with  $p \prec r$  and  $rt = p$ , then  $t$  contains a cycle.

First, if there is no  $r$  such that  $p \prec r$ , we claim that  $t$  is the constant transformation ( $Q \rightarrow p$ ). Consider any state  $q \in Q$  such that  $qt \neq p$ . Then  $p \prec qt$  by Remark 2, contradicting that there is no state  $r$  such that  $p \prec r$ .

So let  $r$  be some state such that  $p \prec r$ . Consider the sequence  $r, rt, rt^2, \dots$ . By Remark 2,  $p \preceq rt^i$  for all  $i \geq 0$ . If  $rt^k = p$  for some  $k \geq 1$ , let  $i$  be the smallest such  $k$ ; we have  $(rt^{i-1})t = p$ , contradicting (ii). Since  $p$  is the only fixed point by (i), we have  $rt^i \neq rt^{i-1}$ . Since there are finitely many states,  $rt^i = rt^j$  for some  $i$  and  $j$  such that  $0 \leq i < j - 1$ , and so the states  $rt^i, rt^{i+1}, \dots, rt^j = rt^i$  form a cycle.

We have shown that for every transformation  $t$  in  $T_n$  there is a corresponding transformation  $f(t)$  in  $S_n$ , and  $f$  is injective. So  $|T_n| \leq |S_n| = n^{n-1} + n - 1$ .  $\square$

Next we prove that  $S_n$  is the only transition semigroup meeting the bound. It follows that minimal DFAs of left ideals with the maximal syntactic complexity have maximal-length chains of length 2.

**Theorem 2.** *If  $T_n$  has size  $n^{n-1} + n - 1$ , then  $T_n = S_n$ .*

*Proof.* Consider a maximal-length chain of states strictly ordered by  $\prec$  in  $\mathcal{D}_n$ . If its length is 2, then by Lemma 2,  $T_n$  is a subsemigroup of  $S_n$ . Thus only  $T_n = S_n$  reaches the bound in this case.

Assume now that the length of a maximal-length chain is at least 3. Then there are states  $p$  and  $r$  such that  $0 \prec p \prec r$ . Let  $R = \{q \mid p \prec q\}$ , and let  $X = Q \setminus (R \cup \{0, p\})$ . We shall show that there exists a transformation  $s$  that is in  $S_n$  but not in  $f(T_n)$ . To define  $s$  we use the constant transformation  $u = (Q \rightarrow p)$  as an auxiliary transformation. Let  $0s = 0$ ,  $ps = r$ ,  $rs = 0$  for all  $r \in R$ , and  $qs = qu = p$  for  $q \in X$ ; these are precisely the rules we used in Case 3(c) in the proof of Theorem 1. By Remark 3,  $s \in S_n$ .

It remains to be shown that there is no transformation  $t \in T_n$  such that  $s = f(t)$ . The proof that  $s$  is different from the transformations  $f(t)$  of Cases 1, 2, 3(a) and 3(b) is exactly the same as the corresponding proof in Case 3(c) following the definition of  $s$ .

It remains to verify that there is no  $u' \in T_n$  in Case 3(c) such that  $f(u') = s$ . Suppose there is such a  $u'$ . Recall that states  $p$  and  $r$  satisfying  $0 \prec p \prec r$  have been fixed by assumption. By the definition of  $s$ , state  $p$  satisfies the conditions (a)  $0 \prec p \prec ps$  and (b)  $ps^2 = 0$ . We claim that  $p$  is the only state satisfying these conditions. Indeed, if  $q \neq p$  then either  $qs = 0$ ,  $q \not\prec qs = 0$  and (a) is violated, or  $qs = p$ ,  $qs^2 = ps = r \neq 0$  and (b) is violated. This observation is used in the proof of Case 3(c) to prove the claim below.

Both  $u$  and  $u'$  satisfy the conditions of Case 3(c), except that  $u$  fails the condition  $u \notin S_n$ . However, that latter condition is not used in the proof that if  $u \neq u'$  and  $u'$  satisfy the other conditions of Case 3(c), then  $s' \neq s$ , where  $s'$  is the transformation obtained from  $u'$  by the rules of  $s$ . Thus  $s$  is also different from the transformations in  $f(T_n)$  from Case 3(c).

Because  $s \notin f(T_n)$ ,  $s \in S_n$  and  $f(T_n) \subseteq S_n$ , the bound  $n^{n-1} + n - 1$  cannot be reached if the length of the maximal-length chains is not 2.  $\square$

## 4 Two-Sided Ideals

If a language  $L$  is a right ideal, then  $L = L\Sigma^*$  and  $L$  has exactly one final quotient, namely  $\Sigma^*$ ; hence this also holds for two-sided ideals. For  $n \geq 3$ , in a two-sided ideal every maximal chain is of length at least 3: it starts with  $L$ , every quotient contains  $L$  and is contained in  $\Sigma^*$ .

It was proved in [3, Theorem 6, p. 125] that the transition semigroup of the following DFA of a two-sided ideal meets the bound  $n^{n-2} + (n-2)2^{n-2} + 1$ .

**Definition 2 (Witness: Two-Sided Ideals).** For  $n \geq 4$ , define the DFA  $\mathcal{W}_n = (Q, \Sigma_{\mathcal{W}}, \delta_{\mathcal{W}}, 0, \{n-1\})$ , where  $Q = \{0, \dots, n-1\}$ ,  $\Sigma_{\mathcal{W}} = \{a, b, c, d, e, f\}$ , and  $\delta_{\mathcal{W}}$  is defined by  $a: (1, 2, \dots, n-2)$ ,  $b: (1, 2)$ ,  $c: (n-2 \rightarrow 1)$ ,  $d: (n-2 \rightarrow 0)$ , for  $q = 0, \dots, n-2$ ,  $\delta(q, e) = 1$  and  $\delta(n-1, e) = n-1$ , and  $f: (1 \rightarrow n-1)$ . For  $n = 4$ , inputs  $a$  and  $b$  coincide, and we can use  $\Sigma_{\mathcal{W}} = \{b, c, d, e, f\}$ .

*Remark 5.* If  $n = 1$ , the only two-sided ideal is  $\Sigma^*$ , its syntactic complexity is 1, and the bound above is not tight. If  $n = 2$ , each two-sided ideal is of the form  $L = \Sigma^* \Gamma \Sigma^*$ , where  $\emptyset \subsetneq \Gamma \subseteq \Sigma$ , its syntactic complexity is 2, and the bound is tight. If  $n = 3$ , there are eight transformations that are initially aperiodic and such that  $(n-1)t = t$  (the property of a right-ideal transformation). We have verified that the DFA having all eight or any seven of the eight transformations is not a two-sided ideal. Hence 6 is an upper bound, and we know from [3] that the transformations  $[1, 2, 2]$ ,  $[0, 0, 2]$ , and  $[0, 1, 2]$  generate a 6-element semigroup. From now on we may assume that  $n \geq 4$ .

We consider a minimal DFA  $\mathcal{D}_n = (Q, \Sigma_{\mathcal{D}}, \delta_{\mathcal{D}}, 0, \{n-1\})$ , where  $Q = \{0, \dots, n-1\}$ , of an arbitrary two-sided ideal with  $n$  quotients, and let  $T_n$  be the transition semigroup of  $\mathcal{D}_n$ . We also deal with the witness DFA  $\mathcal{W}_n = (Q, \Sigma_{\mathcal{W}}, \delta_{\mathcal{W}}, 0, \{n-1\})$  of Definition 2 with transition semigroup  $S_n$ .

*Remark 6.* In  $\mathcal{W}_n$ , the transformations induced by  $a$ ,  $b$ , and  $c$  restricted to  $Q \setminus \{0, n-1\}$  generate all the transformations of the states  $1, \dots, n-2$ . Together with the transformations of  $d$  and  $f$ , they generate all transformations of  $Q$  that fix 0 and  $n-1$ . For any subset  $S \subseteq \{1, \dots, n-2\}$ , there is a transformation—induced by a word  $w_S$ , say—that maps  $S$  to  $n-1$  and fixes  $Q \setminus S$ . Then the words of the form  $w_S e a^i$ , for  $i \in \{0, \dots, n-3\}$ , induce all transformations that maps  $S \cup \{n-1\}$  to  $n-1$  and  $Q \setminus (S \cup \{n-1\})$  to  $i+1$ . In  $\mathcal{W}_n$ , there is also the constant transformation  $ef: (Q \rightarrow n-1)$ .

**Lemma 3.** If  $n \geq 4$  and a maximal-length chain in  $\mathcal{D}_n$  strictly ordered by  $\prec$  has length 3, then  $|T_n| \leq n^{n-2} + (n-2)2^{n-2} + 1$ , and  $T_n$  is a subsemigroup of  $S_n$ .

*Proof.* Consider an arbitrary transformation  $t \in T_n$ ; then  $(n-1)t = n-1$ . If  $0t = 0$ , then any state not in  $\{0, n-1\}$  can possibly be mapped by  $t$  to any one of the  $n$  states; hence there are at most  $n^{n-2}$  such transformations.

If  $0t \neq 0$ , then  $0 \prec 0t$ . Consider any state  $q \notin \{0, 0t\}$ ; since  $\mathcal{D}_n$  is minimal,  $q$  must be reachable from 0 by some transformation  $s$ , that is,  $q = 0s$ . If  $0st \notin \{0t, n-1\}$ , then  $0t \prec 0st$  by Remark 2. But then we have the chain  $0 \prec 0t \prec 0st \prec n-1$  of length 4, contradicting our assumption. Hence we must have either  $0st = 0t$ , or  $0st = n-1$ . For a fixed  $0t$ , a subset of the states in  $Q \setminus \{0, n-1\}$  can be mapped to  $0t$  and the remaining states in  $Q \setminus \{0, n-1\}$  to  $n-1$ , thus giving  $2^{n-2}$  transformations. Since there are  $n-2$  possibilities for  $0t$ , we obtain the second part of the bound. Finally, all states can be mapped to  $n-1$ .

By Remark 6 all of the above-mentioned transformations are in  $S_n$ .  $\square$

**Theorem 3 (Two-Sided Ideals, Factor-Closed Languages).** *If  $L$  is a two-sided ideal or a factor-closed language with  $n \geq 4$  quotients, then its syntactic complexity is less than or equal to  $n^{n-2} + (n-2)2^{n-2} + 1$ .*

*Proof.* It suffices to prove the result for two-sided ideals. As we did for left ideals, we show that  $|T_n| \leq |S_n|$ , by constructing an injective function  $f: T_n \rightarrow S_n$ .

We have  $q \preceq n-1$  for any  $q \in Q$ , and  $n-1$  is a fixed point of every transformation in  $T_n$  and  $S_n$ .

For a transformation  $t \in T_n$ , consider the following cases:

**Case 1:**  $t \in S_n$ .

The proof is the same as that of Case 1 of Theorem 1.

**Case 2:**  $t \notin S_n$ , and  $0t^2 \neq 0t$ .

Let  $0t = p \prec \dots \prec pt^k = pt^{k+1}$  be the chain defined from  $p$ .

• **(a):**  $pt^k \neq n-1$ .

The proof is the same as that of Case 2 of Theorem 1.

• **(b):**  $pt^k = n-1$  and  $k \geq 2$ .

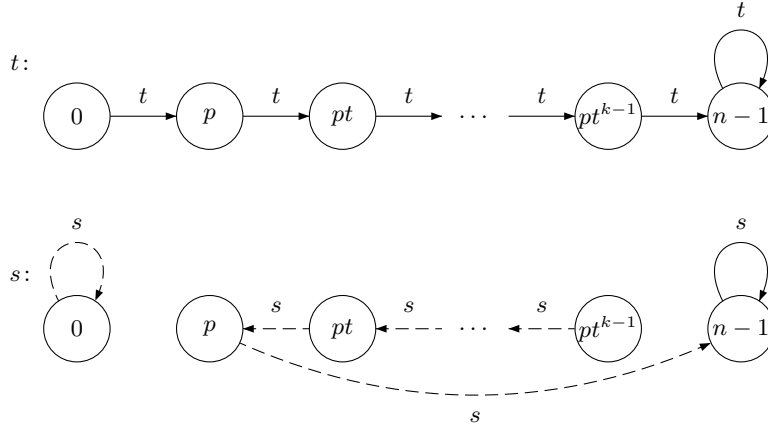
Let  $f(t) = s$ , where  $s$  is the transformation shown in Figure 5 and defined by

$$\begin{aligned} 0s &= 0, & pt^i s &= pt^{i-1} \text{ for } 1 \leq i \leq k-1, & ps &= n-1, \\ qs &= qt \text{ for the other states } q \in Q. \end{aligned}$$

By Remark 6,  $s \in S_n$ . Note that  $s$  contains the cycle  $(p, pt)$  where  $pt \succ p$ ,  $pts = p$  and  $ps = n-1$ . By Proposition 1,  $pts \succeq ps$ , that is,  $p \succeq n-1$ , which contradicts the fact that  $p \neq n-1$ , since  $pt \neq p$ . Thus  $s$  is not in  $T_n$ , and so it is different from the transformations of Case 1.

Observe that  $s$  does not have a cycle with states strictly ordered by  $\prec$ , since no state from  $\{0, p, pt, \dots, pt^{k-1}\}$  can be in a cycle, and  $t$  cannot have such a cycle. Hence  $s$  is different from the transformations of Case 2(a).

In  $s$ , there is a unique state  $q$  such that  $qs = n-1$  and for which there exists a state  $r$  such that  $r \succ q$  and  $rs = q$ , and that this state  $q$  must be  $p$ . Indeed, if  $q \neq p$ , then  $qt = qs = n-1$  by the definition of  $s$ . From  $r \succ q$ , we have  $rt \succeq qt = n-1$ ; hence  $rs = rt = n-1$  and  $rt \neq q$ —a contradiction. Hence  $q = p$ .



**Fig. 5.** Case 2(b) in the proof of Theorem 3.

By a similar argument, we show that there exists a unique state  $q$  such that  $q \succ p$ , and  $qs = p$ , and that this state  $q$  must be  $pt$ . If  $q \neq pt$  then  $qs = qt$ . But  $q \succ qt$  and  $p = qt \succeq qt^2 = pt$  contradicts that  $p \prec pt$ . Continuing in this way for  $pt^2, \dots, pt^{k-1}$  we show that there is a unique chain  $pt^{k-1} \xrightarrow{s} \dots \xrightarrow{s} pt \xrightarrow{s} p$ .

If  $t' \neq t$  is another transformation satisfying the conditions of this case, we define  $s'$  like  $s$ . Now suppose that  $s = f(t) = f(t') = s'$ . Since we have a unique state  $p$  such that  $ps = n-1$  for which there exists a state  $r$  such that  $r \succ p$  and  $rs = p$ , we have  $0t = 0t' = p$ . Also the chain of states  $p, pt, pt^2, \dots, pt^{k-1}$  is unique in  $s$  and  $s'$  as we have shown above; so  $pt^i = pt'^i$  for  $i = 1, \dots, k-1$ . Since the other states are mapped by  $s$  exactly as by  $t$  and  $t'$ , we have  $t = t'$ .

• **(c):**  $pt = n-1$ .

Let  $P = \{0, p, n-1\}$ . Since  $n \geq 4$ , there must be a state  $r \notin P$ . If  $p \prec r$  for all  $r \notin P$ , then  $n-1 = pt \preceq rt$ ; hence  $rt = n-1$  for all such  $r$ , and  $qt \in \{p, n-1\}$  for all  $q \in Q$ . By Remark 6, there is a transformation in  $S_n$  that maps  $S \cup \{n-1\}$  to  $n-1$ , and  $Q \setminus (S \cup \{n-1\})$  to  $p$  for any  $S \subseteq \{1, \dots, n-2\}$ . Thus  $t \in S_n$ —a contradiction.

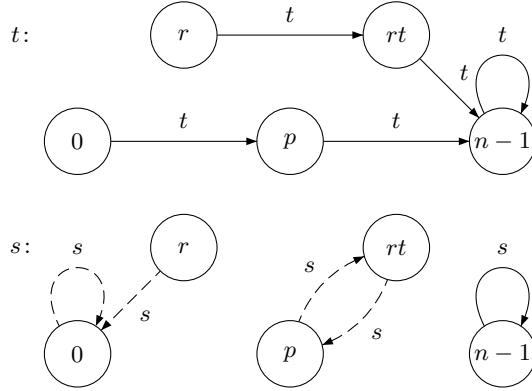
In view of the above, there must exist a state  $r \notin P$  such that  $p \not\prec r$ . By Remark 2, we have  $p \preceq rt$  and of course  $rt \preceq n-1$ . If  $rt$  is  $p$  or  $n-1$  for all  $r \notin P$ , we again have the situation described above, showing that  $t \in S_n$ . Hence there must exist an  $r \notin P$  such that  $p \not\prec r$  and  $p \prec rt \prec n-1$ .

Also we claim that  $t$  does not have a cycle. Indeed, if  $p \preceq q$ , then  $q$  is mapped to  $n-1$ ; if  $p \not\prec q$ , then  $q$  is mapped to a state  $qt \succeq p$  and again  $q$  cannot be in a cycle since the chain starting with  $q$  ends in  $n-1$ .

Let  $f(t) = s$ , where  $s$  is the transformation shown in Figure 6 and defined by

$$\begin{aligned} 0s &= 0, & ps &= rt, & (rt)s &= p, & rs &= 0, \\ qs &= qt \text{ for the other states } q \in Q. \end{aligned}$$

Since  $s$  fixes both 0 and  $n-1$ , it is in  $S_n$  by Remark 6. But  $s$  is not in  $T_n$ , as we have the cycle  $(p, rt)$  with  $p \prec rt$ . So  $s$  is different from the transforma-



**Fig. 6.** Case 2(c) in the proof of Theorem 3.

tions of Case 1. Since  $s$  maps a state other than 0 to 0, it is different from the transformations of Cases 2(a) and 2(b).

Observe that  $t$  does not map any state to 0. Consequently, in  $s$  there is the unique state  $r \neq 0$  mapped to 0. Also, as  $t$  does not contain a cycle, the only cycle in  $s$  must be  $(p, rt)$ .

If  $t' \neq t$  is another transformation satisfying the conditions of this case, we define  $s'$  like  $s$ . Now suppose that  $s = f(t) = f(t') = s'$ . Because both  $s$  and  $s'$  have the unique non-fixed point  $r$  mapped to 0,  $r = r'$ . Also  $s$  and  $s'$  contain the unique cycle  $(p, rt)$ ,  $p \prec rt$ . Thus  $p = p'$ ,  $pt = pt' = n-1$  and  $rt = rt'$ . It follows that  $0t = 0t' = p$ . Because  $p \prec rt = rt'$ , we have  $(rt)t = (rt)t' = n-1$ . The other states are mapped by  $s$  exactly as by  $t$  and  $t'$ , and so  $t = t'$ .

**Case 3:**  $t \notin S_n$ ,  $0t = p \neq 0$  and  $pt = p$ .

- (a):  $t$  has a cycle.

The proof is analogous to that of Case 3(a) in Theorem 1, but we need to ensure that  $s$  is different from the  $s$  of Cases 2(b) and 2(c).

Here there is the state  $r$  such that  $r \prec rs$ , and  $rs$  and  $rs^2$  are not comparable under  $\preceq$ . Consider a transformation  $t'$  that fits in Case 2(b). Then in  $s'$  every state  $q = pt^i$  for  $0 \leq i \leq k-1$ , and  $q = 0$ , is mapped to a state comparable with  $q$  under  $\preceq$ , and the other states are mapped as in  $t'$ . Since  $t' \in T_n$  cannot contain a state  $r'$  such that  $r' \prec r't$  and  $r't$  and  $r't^2$  are not comparable under  $\preceq$ , it follows that  $s'$  also does not contain such a state. Thus  $s \neq s'$ .

For a distinction from the transformations of Case 2(c) observe that  $s$  does not map to 0 any state other than 0.

- (b):  $t$  has no cycles and has a fixed point  $r \notin \{p, n-1\}$ .

The proof is analogous to that of Case 3(b) in Theorem 1, but we need to ensure that  $s$  is different from the  $s$  of Cases 2(b) and 2(c).

Since  $s$  maps to 0 a state other than 0, this case is distinct from Case 2(b). Because  $t$  does not have a cycle, and no state  $q$  mapped to 0 can be in a cy-

cle in  $s$ , it follows that  $s$  does not have a cycle. Thus  $s$  is different from the transformations of Case 2(c).

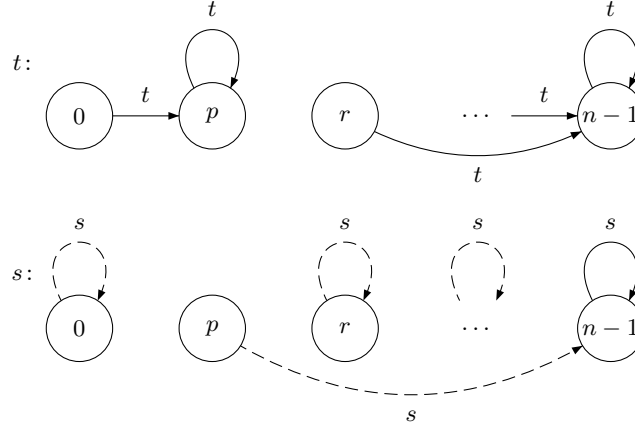
- **(c):**  $t$  has no cycles and no fixed point  $r \notin \{p, n-1\}$ , but has a state  $r \succ p$  mapped to  $p$ .

The proof is analogous to that of Case 3(c) in Theorem 1, but we need to ensure that  $s$  is different from the  $s$  of Cases 2(b) and 2(c).

As before, since  $s$  maps to 0 a state other than 0, this case is distinct from Case 2(b). In  $s$ , 0 cannot be in a cycle, no state  $q \succ p$  mapped to 0 can be in a cycle and  $p$  cannot be in a cycle as  $ps = r$  and  $rs = 0$ . Since the other states are mapped as in  $t$ ,  $s$  does not have a cycle. Thus  $s$  is different from the transformations of Case 2(c).

- **(d):**  $t$  has no cycles, no fixed point  $r \notin \{p, n-1\}$ , and no state  $r \succ p$  mapped to  $p$ , but has a state  $r$  such that  $p \prec r \prec n-1$ , mapped to  $n-1$ . Let  $f(t) = s$ , where  $s$  is the transformation shown in Figure 7 and defined by

$$\begin{aligned} 0s = 0, \quad qs = q \text{ for states } q \text{ such that } qt = n-1, \quad ps = n-1 \\ qs = qt \text{ for the other states } q \in Q. \end{aligned}$$



**Fig. 7.** Case 3(d) in the proof of Theorem 3.

By Remark 6,  $s \in S_n$ . However,  $s$  is not in  $T_n$ , as we have a fixed point  $r$  such that  $p \prec r \prec n-1$  and  $ps = n-1$ . So Proposition 1 yields  $n-1 = ps \preceq rs = r$ —a contradiction. Thus  $s$  is different from the transformations of Case 1.

Transformation  $s$  does not have any cycles, as  $t$  does not have one in this case and fixed points  $q$  and  $p$  cannot be in a cycle. So  $s$  is different from the transformations of Cases 2(a) and 3(a). Also, since  $p$  is the unique state mapped to  $n-1$  and there is no state  $r \succ p$  mapped to  $p$ ,  $s$  is different from the transformations of Case 2(b). For a distinction from the transformations of Cases 2(c), 3(b) and 3(c), observe that  $s$  does not map to 0 any state other than 0.

If  $t' \neq t$  is another transformation satisfying the conditions of this case, we define  $s'$  like  $s$ . Now suppose that  $s = f(t) = f(t') = s'$ . Observe that  $t$  does not

have a fixed point other than  $n - 1$ . So for every fixed point  $q \notin \{0, n - 1\}$  of  $s$  we have  $qt = qt' = n - 1$ . Also, since  $p$  is the unique state mapped to  $n - 1$  in  $s$ ,  $0t = 0t' = p$  and  $pt = pt' = p$ . The other states are mapped by  $s$  as by  $t$  and  $t'$ ; so  $t = t'$ .

• **All cases are covered:**

We need to ensure that any transformation  $t$  fits in at least one case. It is clear that  $t$  fits in Case 1 or 2 or 3. Any transformation from Case 2 fits in Case 2(a) or 2(b) or 2(c). For Case 3, it is sufficient to show that if (i)  $t \notin S_n$  does not contain a fixed point  $r \notin \{p, n - 1\}$ , and (ii) there is no state  $r$ ,  $p \prec r \prec n - 1$ , mapped to  $p$  or  $n - 1$ , then  $t$  has a cycle.

If there is no state  $r$  such that  $p \prec r \prec n - 1$ , then  $qt \in \{p, n - 1\}$  for any  $q \in Q$ , since  $qt \succeq p$ ; by Remark 6,  $t \in S_n$ —a contradiction.

So let  $r$  be some state such that  $p \prec r \prec n - 1$ . Consider the sequence  $r, rt, rt^2, \dots$ . By Remark 2,  $p \preceq rt^i$  for all  $i \geq 0$ . If  $rt^k \in \{p, n - 1\}$  for some  $k \geq 1$ , then let  $i$  be the smallest such  $k$ . Then we have  $(rt^{i-1})t \in p$ , contradicting (ii). Since  $p$  and  $n - 1$  are the only fixed points by (i), we have  $rt^i \neq rt^{i-1}$ . Since there are finitely many states,  $rt^i = rt^j$  for some  $i$  and  $j$  such that  $0 \leq i < j - 1$ , and so the states  $rt^i, rt^{i+1}, \dots, rt^j = rt^i$  form a cycle.  $\square$

**Theorem 4.** *If  $T_n$  has size  $n^{n-2} + (n - 2)2^{n-2} + 1$ , then  $T_n = S_n$ .*

*Proof.* The proof is very similar to that of Theorem 2.

Consider a maximal-length chain of states strictly ordered by  $\prec$  in  $\mathcal{D}_n$ . If its length is 3, then by Lemma 3  $T_n$  is a subsemigroup of  $S_n$ . Thus only  $T_n = S_n$  reaches the bound.

If there is a chain of length 4, then there are states  $p$  and  $r$  such that  $0 \prec p \prec r \prec n - 1$ . Let  $f$  be the injective function from Theorem 3. Consider the transformation  $u$  that maps  $Q \setminus \{n - 1\}$  to  $p$  and fixes  $n - 1$ . Let  $s$  be defined from  $u$  in Case 3(c) of the proof of Theorem 3. The rest of the proof follows the proof of Theorem 2 with Case 3(d) of Theorem 3 added.  $\square$

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